

# Non-Stochastic Uncertainty Quantification of a Multi-Model Response

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- 1 Omnipresent uncertainty
- 2 Non-stochastic methods
  - The worst/best-case scenario method (WBCSM)
  - Fuzzy sets
  - Dempster-Shafer theory (evidence theory, random sets)
- 3 Multimodel UQ
  - Coupling of fuzzy sets and the D-S theory IA
  - Coupling of fuzzy sets and the D-S theory IB
  - Coupling of fuzzy sets and the D-S theory IIA
  - Coupling of fuzzy sets and the D-S theory IIB
- 4 Conclusions

# Uncertain inputs (parameters), “uncertain” (i.e., various) models

## One model

Inputs  $\longrightarrow$  model  $\longrightarrow$  outputs (model response)

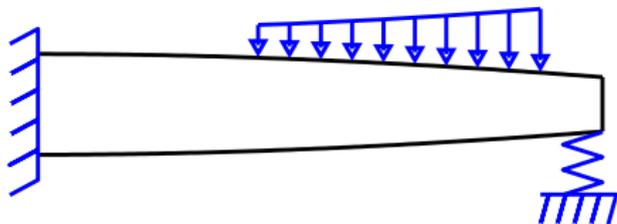
Our mathematical models are burdened with uncertainty in input and, consequently, in output data.

It may happen that a set of relevant models is available, and each model has uncertain inputs.

What UQ approach can we applied to benefit from the availability of a number of models?

# Examples:

A cantilever beam



modeled by the Euler-Bernoulli beam equation (unknown  $v$ )

$$(cEt^3(x)v''(x))'' = f(x) \text{ on } [a, b], \text{ and boundary conditions.}$$

Gao beam, unknown  $v$ :

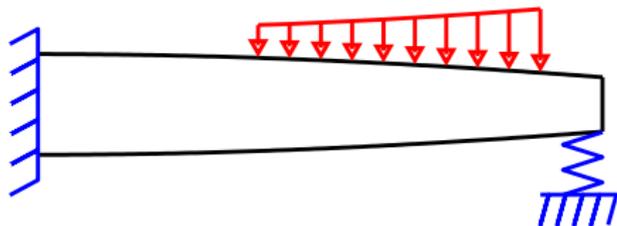
$$(c_1Et^3(x)v''(x))'' - c_2Et(x)(v'(x))^2v''(x) = f(x)$$

Timoshenko beam, unknown  $v, \varphi$ :

$$(cEt^3(x)\varphi')' + \kappa AG(v' - \varphi) = 0, \quad (\kappa AG(v' - \varphi))' = f(x)$$

## Examples:

A cantilever beam where the distributed load  $f$  is an uncertain function.



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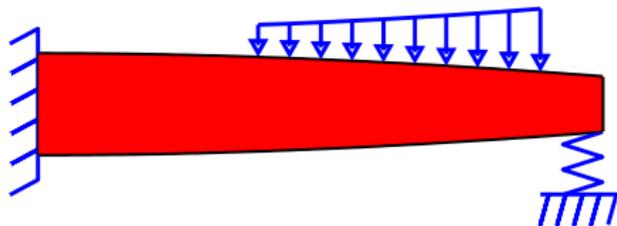
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# Examples:

A cantilever beam where the Young modulus  $E$  and the shear modulus  $G$  are uncertain scalar parameters (or functions).



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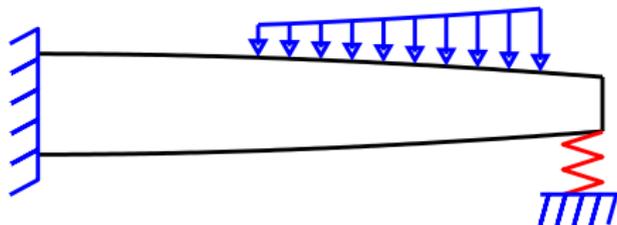
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# Examples:

A cantilever beam where the **boundary condition** (spring stiffness) is represented by an **uncertain scalar parameter**.



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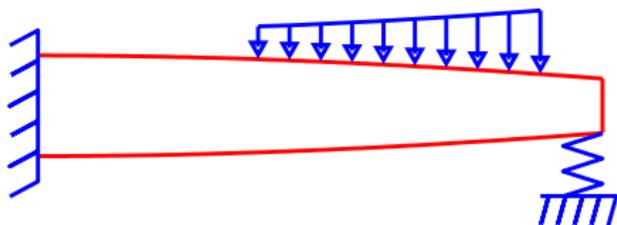
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# Examples:

A cantilever beam where the **beam shape** is determined by an **uncertain function  $t$**  defined on a possibly **uncertain interval  $[a, b]$** .



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# Examples:

Long-term behavior of concrete structures (bridges).

Some widely used creep and shrinkage models:

- ACI 209R (American Concrete Institute)
- AS 3600 Australian Standard Code
- B3 (Bažant, Baweja)
- B4 (RILEM Technical Committee TC-242-MDC; Z.P. Bažant, chair)
- *fib* Model Code 2010 (Fédération Internationale du Béton)
- EuroCode 2 (EU)
- GL2000 (Gardner, Lockman)
- JSCE (Japan Society of Civil Engineers)

# Basic settings (one model problem)

$\mathcal{U}_{\text{ad}}$  ... set of admissible values of uncertain parameters (convex and compact  $\mathcal{U}_{\text{ad}}$  is preferred)

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## Examples

$\mathcal{U}_{\text{ad}}$  is a bounded and closed interval if a scalar parameter is uncertain.

$\mathcal{U}_{\text{ad}}$  is a compact convex subset of  $\mathbb{R}^n$  if an  $n$ -dimensional vector is uncertain.

$\mathcal{U}_{\text{ad}}$  is a compact convex set of functions if a function is uncertain.

# Basic settings (one model problem)

$\mathcal{U}_{\text{ad}}$  ... set of admissible values of uncertain parameters (convex and compact  $\mathcal{U}_{\text{ad}}$  is preferred)

$D(a)u = f$  ... **state problem** dependent on  $a \in \mathcal{U}_{\text{ad}}$

Consequently, its solution  $u \equiv u(a)$  also depends on  $a \in \mathcal{U}_{\text{ad}}$ .

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### Examples

A boundary value problem for an ordinary or a partial differential equation dependent on  $a$ .

An initial value problem dependent on  $a$ .

A variational inequality dependent on  $a$  (then “=” is inappropriate).

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## Examples

Local displacement, temperature, mechanical stress or stress invariants, creep, weight, concentration of chemicals, velocity, etc.

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Assumptions:

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**Remark:** In some problems (like long-term models of concrete),  $\Psi$  is a direct map from  $\mathcal{U}_{\text{ad}}$  to  $\mathbb{R}$ , i.e., a function of several variables (no embedded problem  $D(a)u = f$ ).

# Stochastic methods . . .

. . . are the most popular in uncertainty quantification.

**Input parameters:** random variables, vectors, functions, etc.

**Goal:** To characterize the randomness of the model response and/or the quantity of interest.

**Remark:** It might be difficult to characterize the randomness of input data (insufficient data, inappropriate assumptions about probability distributions, etc.). Other approaches available.

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# The worst/best-case scenario method (WBCSM)

**Goal:** Determine the range the criterion-functional  $\Psi$  over  $\mathcal{U}_{\text{ad}}$ .

To determine the “worst” case scenario (anti-optimization (Elishakoff)), we maximize  $\Psi$  by searching for

$$a^0 = \arg \max_{a \in \mathcal{U}_{\text{ad}}} \Psi(a).$$

If also the “best” case scenario  $a_0 = \arg \min_{a \in \mathcal{U}_{\text{ad}}} \Psi(a)$  is found, then the

**range** of  $\Psi|_{\mathcal{U}_{\text{ad}}}$  is given by

$$I_{\Psi} = [\Psi(a_0), \Psi(a^0)].$$

Owing to the compactness of  $\mathcal{U}_{\text{ad}}$  and the continuity of  $\Psi$ , the interval  $I_{\Psi}$  exists.

**WVCSM will be a key tool in the multimodel UQ.**

# Approximated WBCSM

- Approximation of  $\mathcal{U}_{\text{ad}}$ :

- If  $\mathcal{U}_{\text{ad}} \subset \mathbb{R}^n$ , then usually  $\mathcal{U}_{\text{ad}}^{\text{app}} \equiv \mathcal{U}_{\text{ad}}$  (no approximation).
- If  $\mathcal{U}_{\text{ad}}$  is a subset of a function space, then  $\mathcal{U}_{\text{ad}}$  is approximated through  $\mathcal{U}_{\text{ad}}^{\text{app}} \subset \mathbb{R}^n$  (piecewise linears, polynomials, splines, etc.).

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- **Approximation  $u_h$  of the state solution  $u$  by a numerical method** (finite differences, finite elements, boundary elements, etc.)

The **approximate worst/best-case scenario problem**: Find

$$a_h^{\text{app},0} = \arg \max_{a^{\text{app}} \in \mathcal{U}_{\text{ad}}^{\text{app}}} \Psi(a^{\text{app}}), \quad a_{0,h}^{\text{app}} = \arg \min_{a^{\text{app}} \in \mathcal{U}_{\text{ad}}^{\text{app}}} \Psi(a^{\text{app}}).$$

To obtain the range of  $\Psi(a^{\text{app}})$

$$I_{\Psi}^{\text{app},h} = \left[ \Psi \left( a_{0,h}^{\text{app}} \right), \Psi \left( a_h^{\text{app},0} \right) \right],$$

optimization tools are needed:

- global methods

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## Examples

- stochastic (based on the Monte Carlo method, for instance)
- metaheuristic
  - simulated annealing
  - swarm based algorithms
  - ant colony optimization
  - evolutionary algorithms (genetic algorithms, evolution strategies)
- and others

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## Examples

- derivative free methods
- gradient methods (sensitivity analysis, automatic differentiation)

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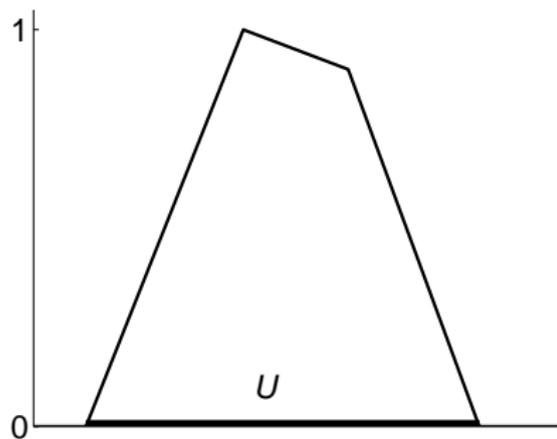
**Remarks:** Coordinate-wise monotonicity of  $\Psi$  is beneficial.  
A sampling-based rough approximation of  $I_{\Psi}^{\text{app},h}$  is sometimes sufficient in practice.

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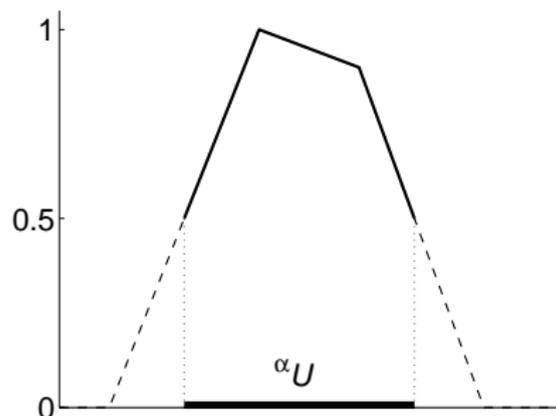
# Fuzzy set theory (Lotfi Aliasker Zadeh, 1921–2017)

Initially a quantification of linguistic variables (“small”, “average”, “large”, etc.); a calculus based on the **membership function**  $\mu_U$



of a set  $U$ . The value  $\mu_U(x)$  represents the degree to which  $x$  belongs to the set  $U$ . The higher the value, the stronger the membership. For our purposes, the set  $U$  is compact and convex,  $\mu_U$  is continuous and concave on  $U$ ,  $\mu_U : U \rightarrow [0, 1]$ .

# Fuzzy set theory: $\alpha$ -cut



For  $\alpha \in (0, 1]$ , a subset  ${}^\alpha U$  comprising all  $x \in X$  such that  $\mu_U(x) \geq \alpha$  is called the  $\alpha$ -cut.

It will be convenient to have  $U \equiv {}^0 U = \text{supp}(\mu_U)$ .

We will see later fuzzy sets of vectors or functions.

# Zadeh's extension principle

**Goal:** To infer the membership function of a quantity of interest if the model inputs are fuzzy sets.

**Zadeh's extension principle:** Let  $U$  be fuzzy,  $\Psi : U \rightarrow \mathbb{R}$ , and the range of  $\Psi$  be an interval  $I_\Psi$ . Its membership function  $\mu_{I_\Psi}$  is defined for each  $y \in I_\Psi$  as follows

$$\mu_{I_\Psi}(y) = \max\{\mu_U(a) : a \in U \wedge \Psi(a) = y\}. \quad (1)$$

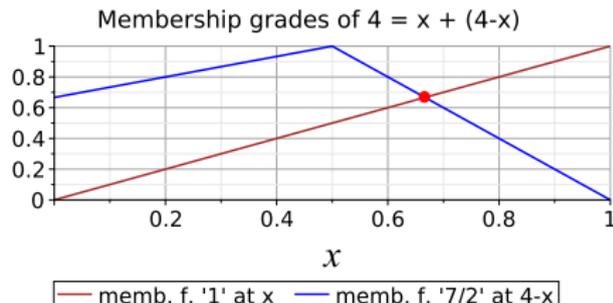
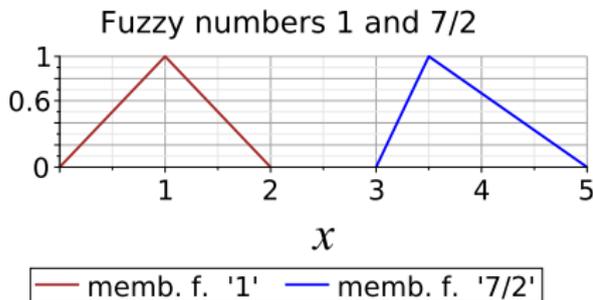
**Generalization:**  $\Psi : \mathcal{U} \rightarrow \mathbb{R}$  where  $\mathcal{U} = U_1 \times U_2 \times \dots \times U_n$ , each  $U_i$  is a fuzzy set, and  $I_\Psi$  is the range of  $\Psi$ . Then, for each  $y \in I_\Psi$ ,

$$\mu_{I_\Psi}(y) = \max\{\min\{\mu_{U_1}(a_1), \dots, \mu_{U_n}(a_n)\} : \bar{a} \in \mathcal{U} \wedge \Psi(\bar{a}) = y\} \quad (2)$$

where  $\bar{a} = (a_1, a_2, \dots, a_n)$ .

**Remarks:** By virtue of our assumptions, the maximum can be used instead of a supremum in (1) and (2). The rule (2) is not the only option, see  $t$ -norms.

## Example (Fuzzy 1 plus fuzzy 7/2, Part I)



$U_1 = [0, 2]$ ,  $U_2 = [3, 5] \Rightarrow$  the range of  $\Psi = U_1 + U_2$  is equal to  $[3, 7]$ .  
 Right: The blue curve represents  $\mu_{U_2}(4 - x)$ .

$$\mu_{I_\Psi}(4) = \max_{x \in [0,1]} \{ \min \{ \mu_{U_1}(x), \mu_{U_2}(4 - x) \} \} = \frac{2}{3} \text{ is attained}$$

for  $\frac{2}{3} \in U_1$  and  $\frac{10}{3} \in U_2$ .

**Remark:** The principle in the form (1) or (2) is not suitable for calculations. An  $\alpha$ -cut technique is a better option.

# Fuzzy set theory and the WBCSM

Let  $\mathcal{U}_{\text{ad}} \equiv U$  and  $\mu_{\mathcal{U}_{\text{ad}}}$  be given. Let  ${}^\alpha\mathcal{U}_{\text{ad}}$  be the  $\alpha$ -cut of  $\mathcal{U}_{\text{ad}}$ ,  $\alpha \in [0, 1]$ .

It can be proved: For  ${}^\alpha\mathcal{U}_{\text{ad}}$ , the **best** ( $a_{0,\alpha}$ , **minimizer**) and **worst** ( $a^{0,\alpha}$ , **maximizer**) scenarios determine

$${}^\alpha I_\Psi = [\Psi(a_{0,\alpha}), \Psi(a^{0,\alpha})]. \quad (3)$$

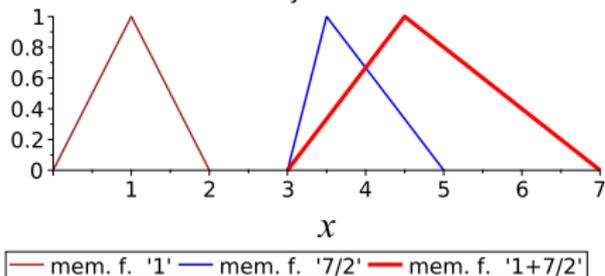
These  ${}^\alpha I_\Psi$  are the  $\alpha$ -cuts of  $I_\Psi \equiv {}^0 I_\Psi = \{\Psi(a) \mid a \in \mathcal{U}_{\text{ad}}\}$ , the fuzzy range of  $\Psi$ .

Then the **membership function**  $\mu_{I_\Psi}$  **can be constructed** via

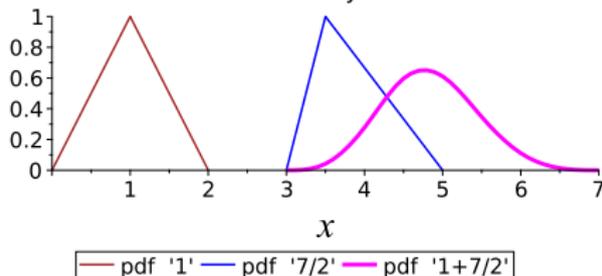
$$\mu_{I_\Psi}(y) = \max\{\alpha \mid y \in {}^\alpha I_\Psi\}, \quad y \in I_\Psi.$$

## Example (Fuzzy 1 plus fuzzy 7/2, Part II)

Fuzzy sets



Probability



Left: By considering  $\alpha$ -cuts of fuzzy 1 and fuzzy 7/2, and by applying the WBCSM (3), we easily obtain the membership function of the sum.

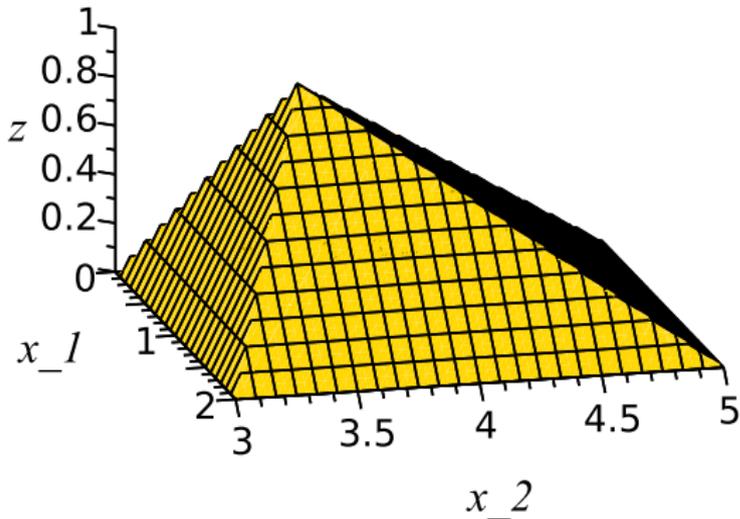
Right: Independent continuous random variables  $X$  and  $Y$  with the respective pdf  $f(x)$  and  $g(y)$ . Then the pdf of  $Z = X + Y$  is

$$q(z) = \int_{-\infty}^{\infty} f(x)g(z-x) dx = \int_{-\infty}^{\infty} f(z-y)g(y) dy$$

# Cartesian product of fuzzy sets

The membership function  $\mu_{U_1 \times U_2}$  of the Cartesian product of two mutually independent fuzzy sets  $U_1$  and  $U_2$  with the respective membership functions  $\mu_{U_1}$  and  $\mu_{U_2}$  is defined by (a common instance of  $t$ -norms)

$$\mu_{U_1 \times U_2}(a_1, a_2) = \min\{\mu_{U_1}(a_1), \mu_{U_2}(a_2)\}, \quad (a_1, a_2) \in U_1 \times U_2.$$



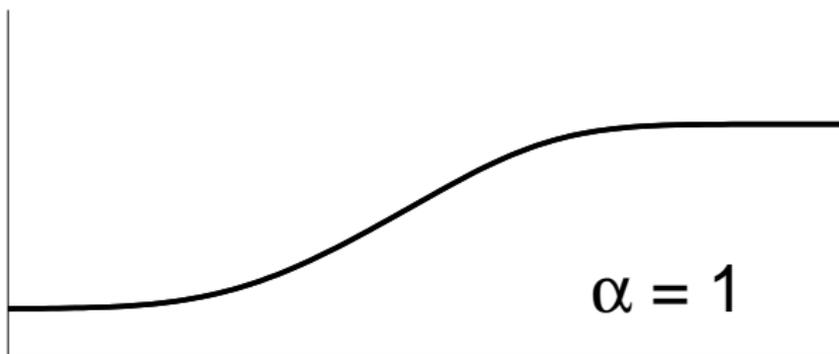
# Fuzzy sets of functions

A parallel to a triangular membership function.

If  $\alpha = 1$ , then  ${}^\alpha \mathcal{U}_{\text{ad}} = \{f_{\text{control}} \in C_K^{(0),1}([a, b])\}$  where  $C_K^{(0),1}([a, b])$  stands for Lipschitz-continuous functions with a Lipschitz constant bounded by  $K > 0$  (functions with a bounded slope).

If  $\alpha \in [0, 1]$  and  $c > 0$ , then

$${}^\alpha \mathcal{U}_{\text{ad}} = \left\{ f \in C_K^{(0),1}([a, b]) : \|f - f_{\text{control}}\|_{C([a,b])} \leq c(1 - \alpha) \right\}$$



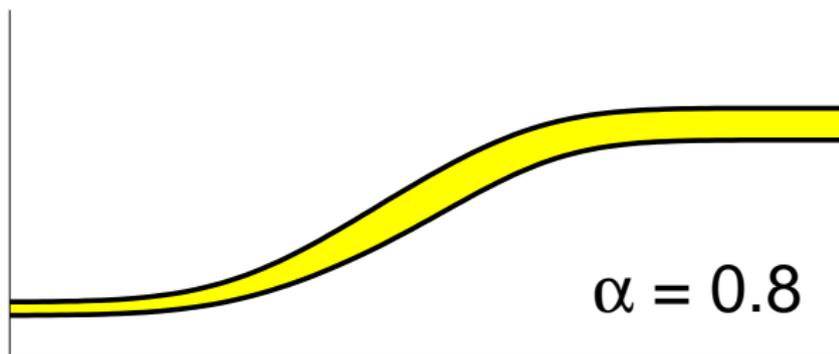
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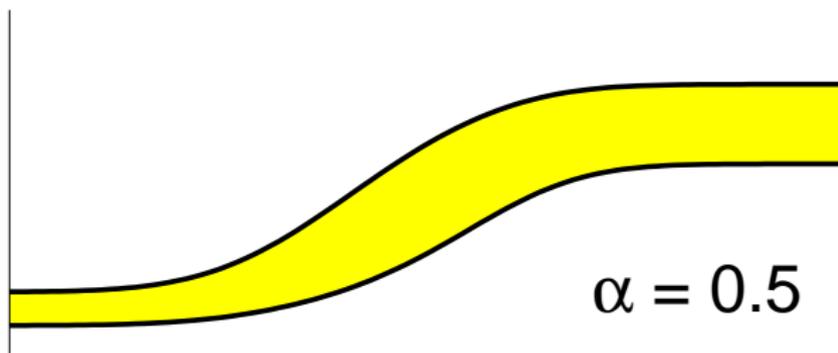
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$${}^\alpha \mathcal{U}_{\text{ad}} = \left\{ f \in C_K^{(0),1}([a, b]) : \|f - f_{\text{control}}\|_{C([a, b])} \leq c(1 - \alpha) \right\}$$



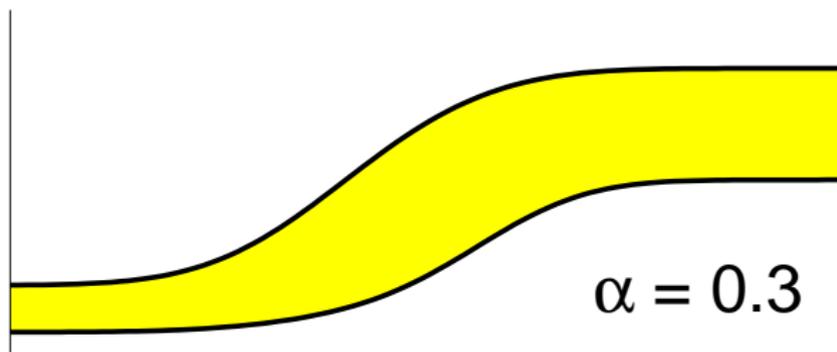
# Fuzzy sets of functions

A parallel to a triangular membership function.

If  $\alpha = 1$ , then  ${}^\alpha \mathcal{U}_{\text{ad}} = \{f_{\text{control}} \in C_K^{(0),1}([a, b])\}$  where  $C_K^{(0),1}([a, b])$  stands for Lipschitz-continuous functions with a Lipschitz constant bounded by  $K > 0$  (functions with a bounded slope).

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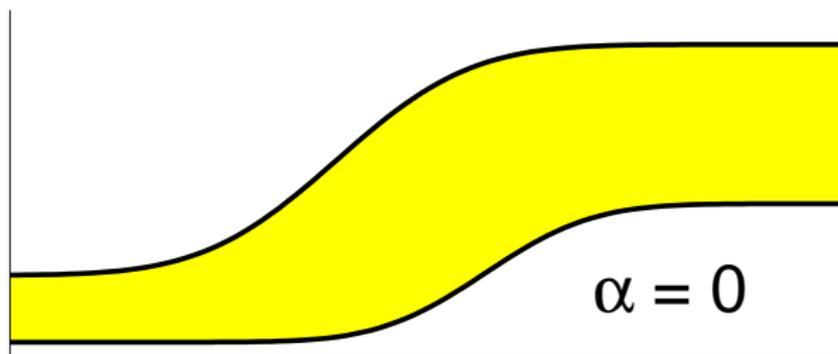
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  - **Dempster-Shafer theory (evidence theory, random sets)**
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# Dempster-Shafer (evidence) theory, random sets

Arthur P. Dempster (\* 1929), Glenn Shafer (\* 1946)

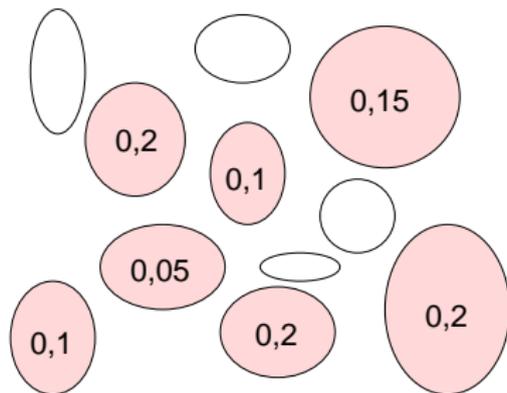
$X$  ... universal set

$P_X$  ... power set of  $X$ , i.e., the set of all subsets of  $X$ .

$m : P_X \rightarrow [0, 1]$  ... basic probability assignment (BPA)

It must satisfy  $m(\emptyset) = 0$  and  $\sum_{\text{all } A \in P_X} m(A) = 1$ .

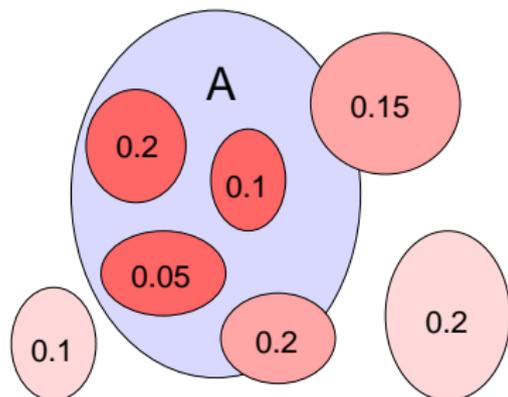
Let  $m(A_i) > 0$  only for a finite number  $N_{\text{BPA}}$  of sets  $A_i \in P_X$ ;  
 $A_i$  are called **focal elements**.



### Definition (*Belief and Plausibility*)

The belief measure of  $A \in P_X$ :  $Bel(A) = \sum_{\text{all } A_i \subseteq A} m(A_i)$ .

The plausibility measure of  $A \in P_X$ :  $Pl(A) = \sum_{\text{all } A_i \cap A \neq \emptyset} m(A_i)$ .



$$Bel(A) = 0.2 + 0.1 + 0.05 = 0.35$$

$$Pl(A) = 0.2 + 0.1 + 0.05 \\ + 0.15 + 0.2 = 0.7$$

# Dempster-Shafer theory and the WBCSM

Let  $A_i$ , where  $i = 1, \dots, N_{\text{BPA}}$ , be both the focal elements of  $m$  and admissible sets.

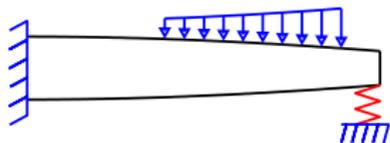
Let  $\Psi(A_i)$ , the range of  $\Psi|_{A_i}$ , be **calculated via the worst-case and best-case scenarios**.

**Extension principle:**  $l_k \in \{\Psi(A_1), \Psi(A_2), \dots, \Psi(A_{N_{\text{BPA}}})\}$  is a focal element of the probability assignment  $m_\Psi$  defined by

$$m_\Psi(l_k) = \sum_{\{i: l_k = \Psi(A_i)\}} m(A_i), \quad k = 1, \dots, M.$$

in the range of  $\Psi$ .

## Example (Uncertain spring stiffness)



Five intervals  $A_1, \dots, A_5$  of the spring stiffness values obtained in laboratory tests.

Their respective credibility degrees (i.e., basic probability assignment) are equal to 0.1, 0.4, 0.1, 0.25, and 0.15.

$\Psi$  stands for the beam tip displacement.

Let  $I_k = \Psi(A_k)$ ,  $k = 1, 2, \dots, 5$ . It is calculated that

$$\begin{aligned}
 I_1 &= [77, 80], m_\Psi(I_1) = 0.1; & I_2 &= [69, 74], m_\Psi(I_2) = 0.4; \\
 I_3 &= [73, 79], m_\Psi(I_3) = 0.1; & I_4 &= [71, 78], m_\Psi(I_4) = 0.25; \\
 I_5 &= [76, 83], m_\Psi(I_5) = 0.15.
 \end{aligned}$$

### Example (Uncertain spring stiffness (cont.))

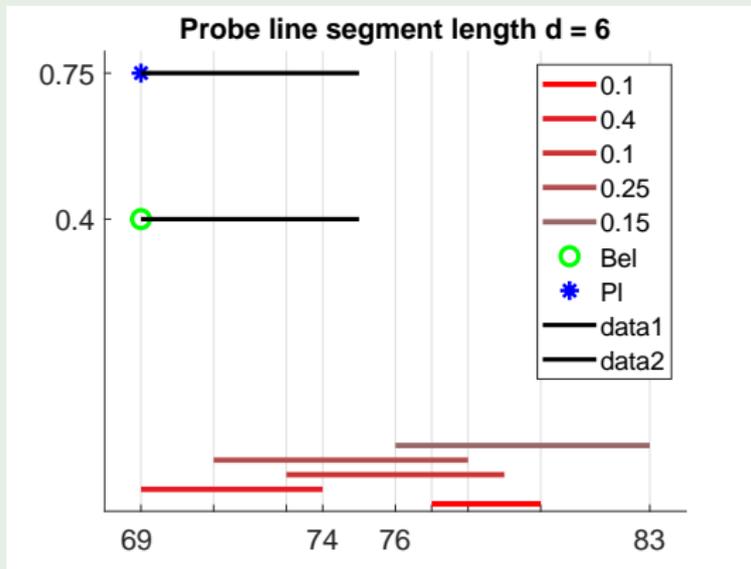
To analyze the (uncertain) behavior of the quantity of interest, let us graph

$$Bel([x, x + d]) \text{ and } Pl([x, x + d]),$$

where  $d \in \{6, 7, 8, 9\}$  is fixed and  $x \in [60, 90]$ .

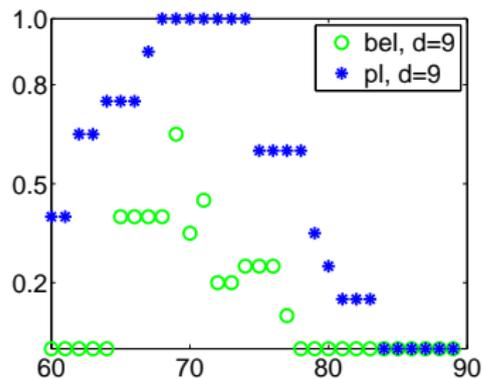
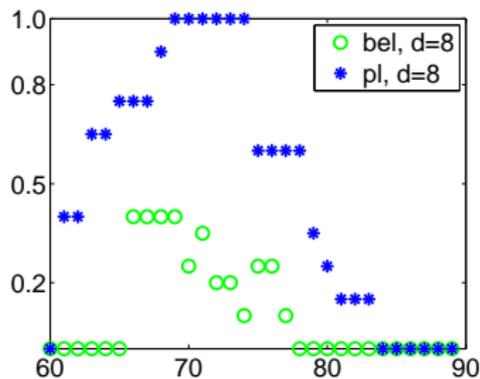
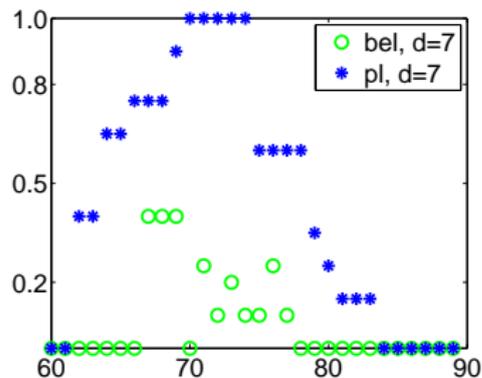
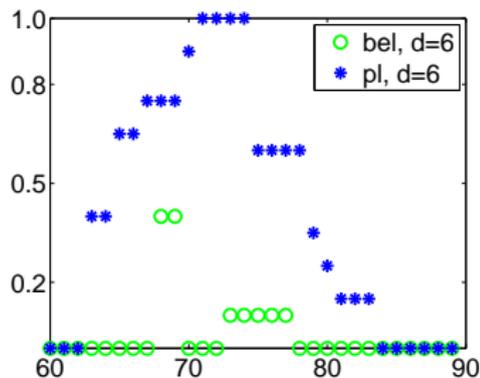
In other words, the “credibility” of chosen intervals  $[x, x + d]$  will be assessed by the evidence we have about the behavior of  $\Psi$ .

## Example (Uncertain spring stiffness (cont.))



$$Bel([69,75])=0.4, \quad Pl([69,75])=0.75$$

# Example (Uncertain spring stiffness: *Bel* and *Pl* results)



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# Coupling of fuzzy sets and the D-S theory IA

Let us have  $M$  models with their fuzzy admissible sets  $\mathcal{U}_{\text{ad}}^i$ ,  $i = 1, \dots, M$ , and with their positive “credibility” weights  $m_i$ ,  $\sum_{i=1}^m m_i = 1$ ; these will serve as the BPA.

One quantity of interest, but observed through  $M$  models, i.e., outputs  $\Psi_i$  are considered.

## Method IA

- 1 Fix  $\alpha \in (0, 1]$  and infer the  $\alpha$ -cuts of fuzzy input data.
- 2 Calculate the output  $\alpha$ -cuts, use the BPA, and define them as focal elements.
- 3 Define a set of intervals and calculate their *Bel* and *Pl* value.

An “industrial” example follows.

# Long-term behavior of concrete (with Lenka Dohnalová)

Let us apply the multimodel UQ to four models

- Model B3 (Bažant, Baweja)
- *fib* Model Code 2010 (Fédération Internationale du Béton)
- EuroCode 2 (EU, CR)
- ACI 209R (American Concrete Institute)

and to a shrinkage strain  $\varepsilon_{sh}$  evolution as the quantity of interest.

For brevity, only Model B3 will be presented in more detail.

# Model B3 by Z. P. Bažant et al. (1995, 2010, USA)

Zdeněk P. Bažant (\* 1937)

In Model B3, strain  $\varepsilon$  evolves in time  $t$  (all times in days) as

$$\varepsilon(t) = J(t; t', t_0)\sigma + \varepsilon_{sh}(t; t_0) + \text{a thermal term,}$$

where  $J$  is the compliance function,  $\sigma$  is uniaxial stress,  $\varepsilon_{sh}$  is the shrinkage strain,  $t'$  is the age at loading,  $t_0$  is the age when drying begins.

Similarly in the other models.

We will limit ourselves to only  $\varepsilon_{sh}$ .

# Shrinkage strain $\varepsilon_{sh}$ in Model B3

$$\varepsilon_{sh}^{B3,p_1}(t) = \frac{c_1 \left( \frac{c_2 w^{2.1}}{f_c^{0.28}} + c_3 \right) (1 - h_{RH}^3) \tanh \left( c_4 \sqrt{\frac{(t - t_0) t_0^{2/25} f_c^{1/4}}{D^2}} \right)}{t_0^{27/25} f_c^{1/4} + c_5 D^2},$$

$$\frac{4 t_0^{2/25} f_c^{1/4} + c_6 t_0^{27/25} f_c^{1/4} + c_7 D^2}{}$$

where the vector  $p_1 = (D, f_c, h_{RH}, w)$  will be considered **uncertain**

- $w$  water content of concrete
- $f_c$  28 day standard cylinder compression strength of concrete
- $h_{RH}$  relative humidity
- $D$  effective cross-section thickness (related to volume to surface ratio of the concrete member)
- $c_1, \dots, c_7$  known constants
- $t_0$  age when drying begins

## Example

### Crisp data:

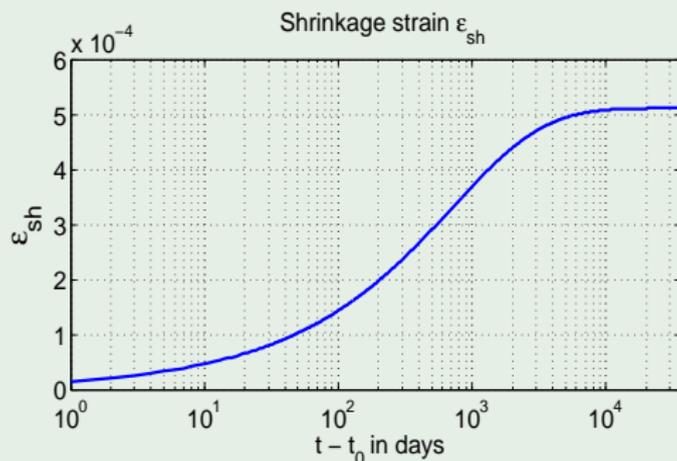
$w = 205$  water content in  $\text{kg/m}^3$

$f_c = 33.3$  mean compression strength in MPa

$h_{RH} = 0.70$  relative humidity

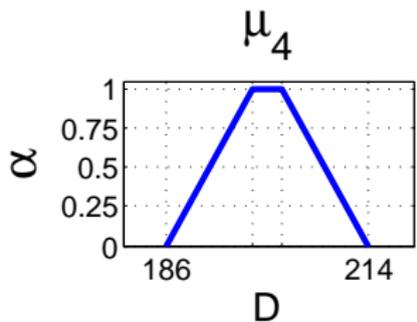
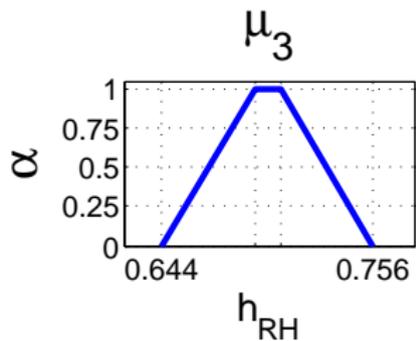
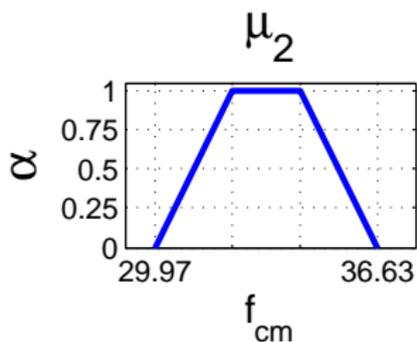
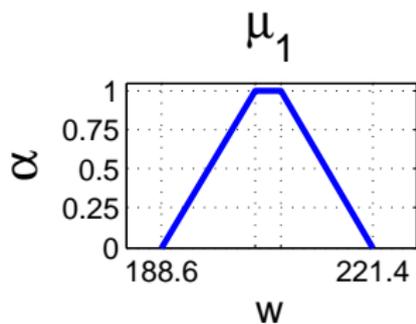
$D = 200$  thickness of the specimen (a concrete slab) in mm

$t_0 = t'_0 = 7$  load and drying in effect beginning day 7

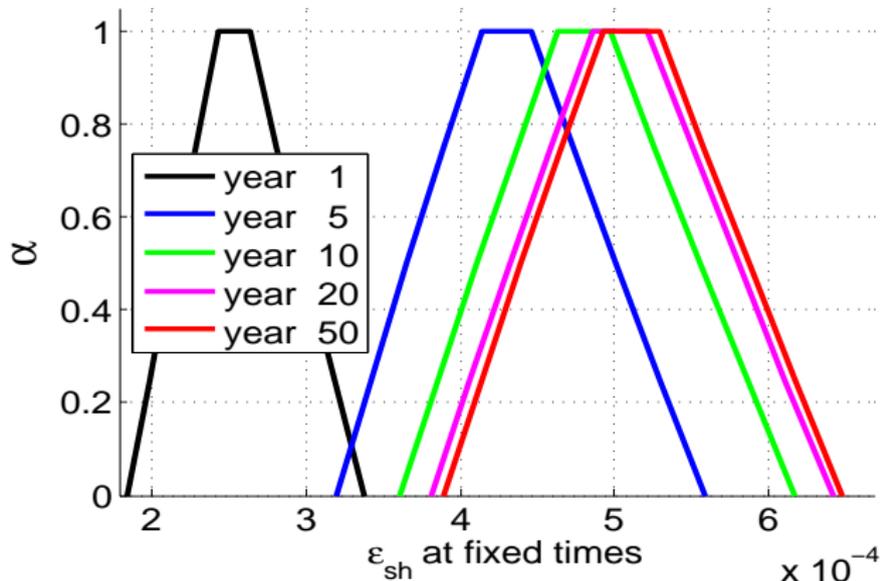


# Model B3; fuzzy inputs

Four fuzzy parameters in Model B3:



# Model B3; output membership functions at fixed $t$



The search for the worst/best-case scenario was facilitated by monotone dependencies of the QoI on uncertain parameters.

# The other strain functions

*fib* Model Code 2010:  $\varepsilon_{sh}^{fib,p_2}(t)$  depends on three fuzzy parameters  $p_2 = (D, f_c, h_{RH})$  and is given by an algebraic formula.

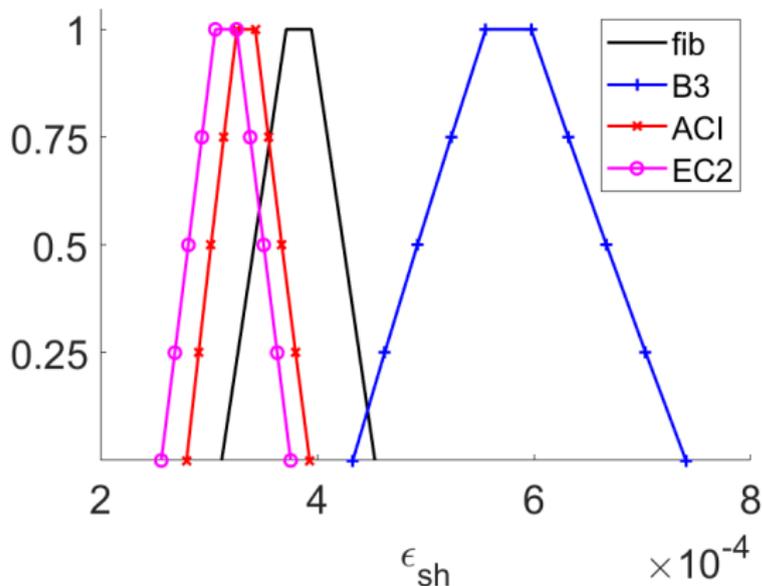
Eurocode 2:  $\varepsilon_{sh}^{EC,p_3}(t)$  depends on the same fuzzy parameters  $p_3 = (D, f_c, h_{RH})$  and is given by an algebraic formula.

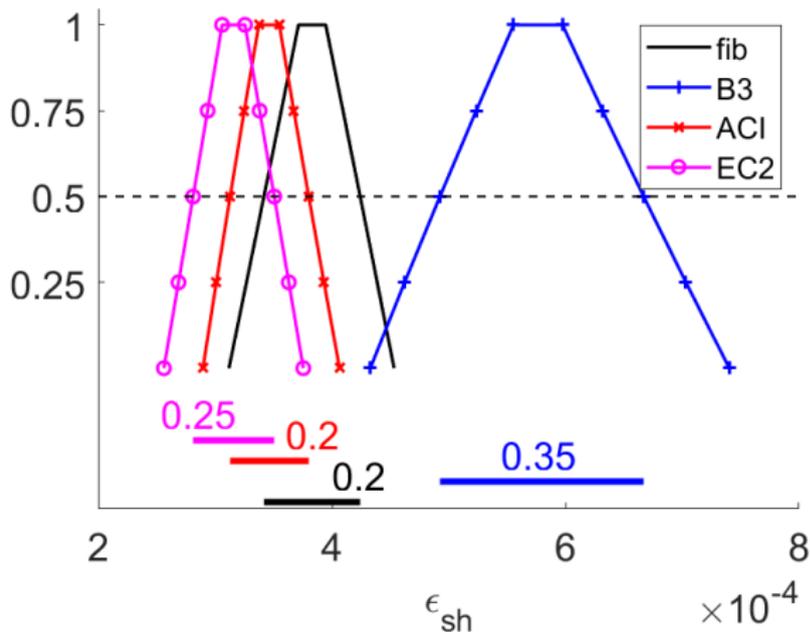
ACI 209R model:  $\varepsilon_{sh}^{ACI,p_4}(t)$  depends on five fuzzy parameters  $p_4 = (D, h_{RH}, \cdot, \cdot, \cdot)$  and is also given by an algebraic formula.

# Coupling with the D-S theory

Up to now, only the B3 model was fuzzified. Let the *fib* Model Code 2010, Eurocode 2, and ACI code be fuzzified, too.

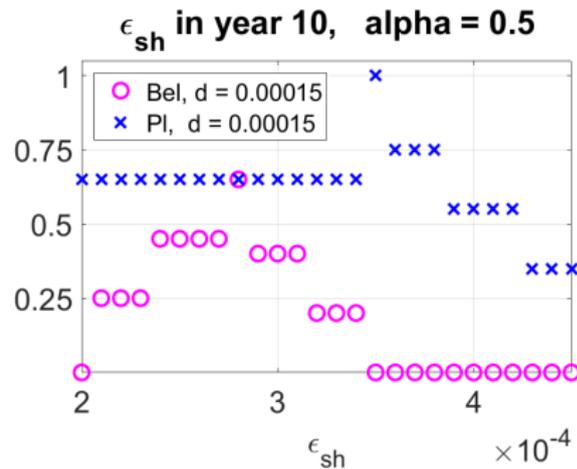
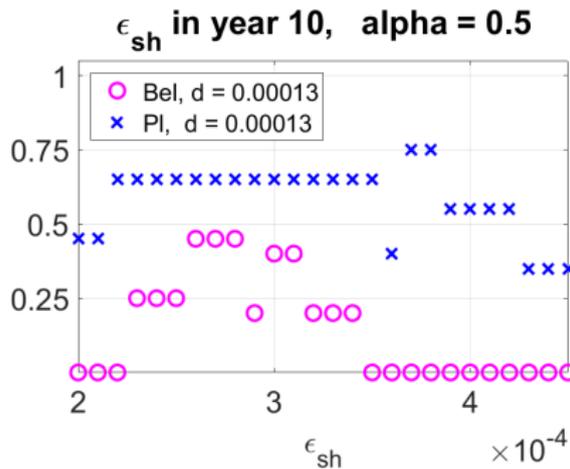
At a fixed time, say,  $t_{\text{fix}} = 3650$  days, we obtain four membership functions of  $\epsilon_{\text{sh}}$ .





Let  $\alpha = 0.5$ , then we use four  $\alpha$ -cuts as focal elements with chosen basic probability assignment 0.2, 0.35, 0.2, and 0.25 for the *fib*, B3, ACI, and Eurocode 2 model, respectively.

Finally, *Belief* and *Plausibility* are calculated for various intervals.



Shrinkage strain  $\epsilon_d$ . Values  $Bel^{0.5}$  and  $PI^{0.5}$  for two sequences of probing intervals.

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# Coupling of fuzzy sets and the D-S theory IB

Recall the plausibility measure of a set  $A$ :  $Pl(A) = \sum_{\text{all } A_i \cap A \neq \emptyset} m(A_i)$ .

The measure of the intersection is not taken into account!

Modification:

$$Pl_{\text{ratio}}(A) = \sum_{\text{all } A_i \cap A \neq \emptyset} m(A_i) \frac{\text{meas } A_i \cap A}{\text{meas } A_i}. \quad (4)$$

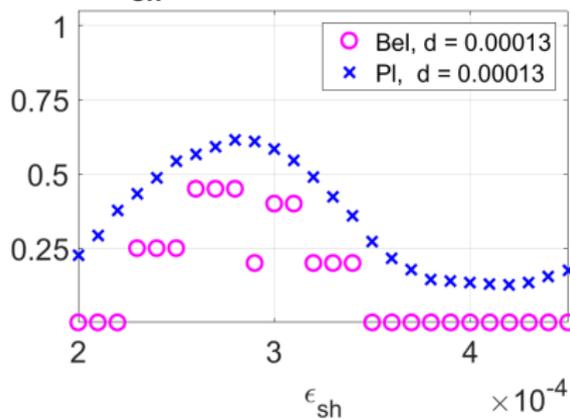
The ratio shows how much of  $A_i$  is covered by  $A$ .

The ratio is equal to one  $\Leftrightarrow A_i \subset A$  (if we neglect troubles with sets of measure zero).

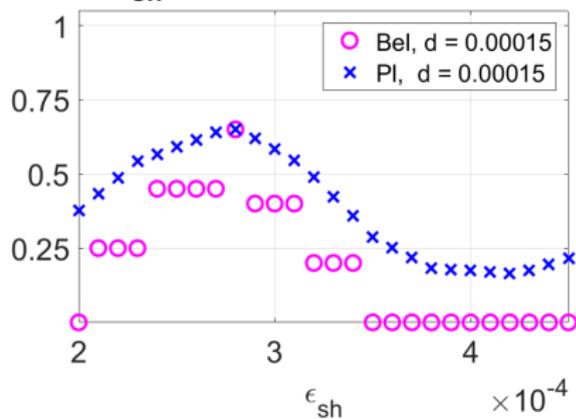
Method IB: The same algorithm as in the Method IA except for (4).

Shrinkage strain  $\epsilon_{sh}$ . Values  $Bel^{0.5}$  and  $Pl^{0.5}$  ratio.

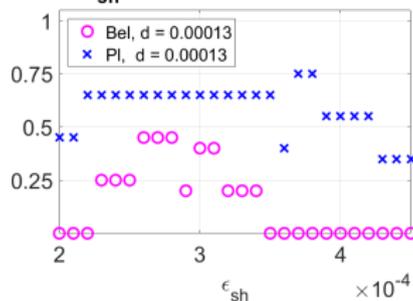
$\epsilon_{sh}$  in year 10, alpha = 0.5



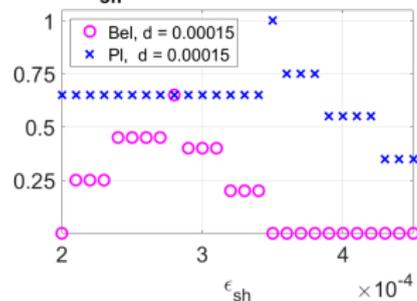
$\epsilon_{sh}$  in year 10, alpha = 0.5



$\epsilon_{sh}$  in year 10, alpha = 0.5



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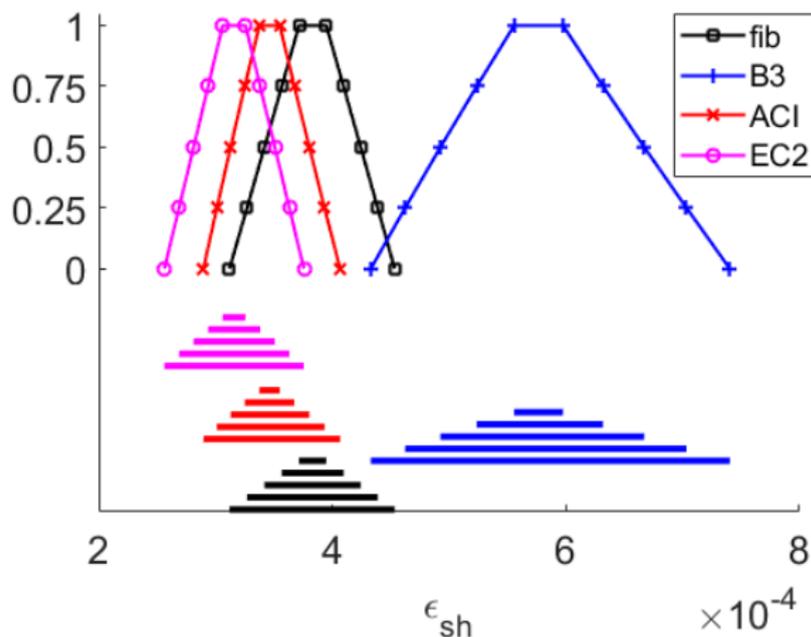


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# Coupling of fuzzy sets and the D-S theory IIA

We can relax the emphasis on one fixed  $\alpha$ -cut. Instead, more (consonant) focal elements can be defined by a set of  $\alpha$ -cuts.



The weights  $w_i$ ,  $i = 0, \dots, 3$ , of the *fib*, B3, ACI, and EC 2 models are 0.2, 0.35, 0.2, and 0.25, respectively.

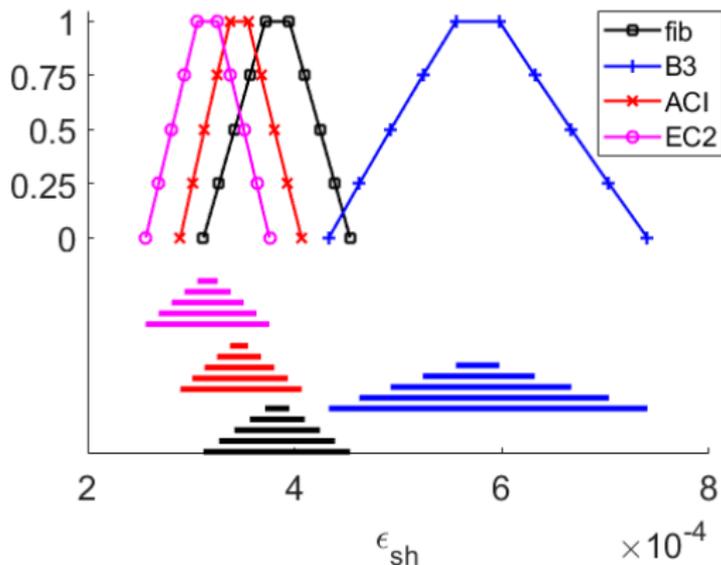
The focal elements  $I_k$  are formed by the  $\alpha$ -cuts  $\alpha_{\epsilon_{sh}}^{fib}$ ,  $\alpha_{\epsilon_{sh}}^{B3}$ ,  $\alpha_{\epsilon_{sh}}^{ACI}$ , and  $\alpha_{\epsilon_{sh}}^{EC2}$ , where  $\alpha = 0, 0.25, 0.5, 0.75, 1$ .

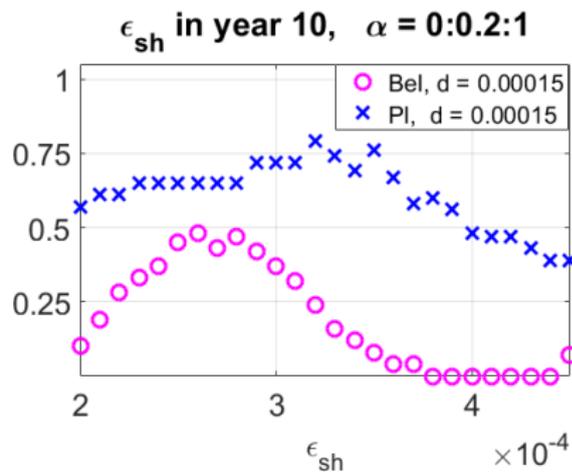
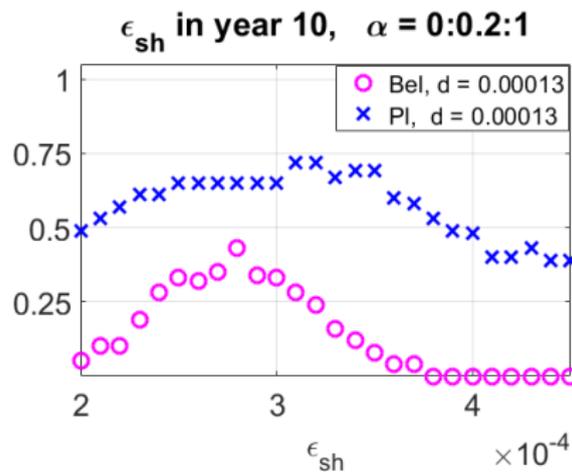
The BPA is defined as

$$m(I_k) = \frac{w_{\lfloor (k-1)/5 \rfloor}}{5},$$

where  $k = 1, 2, \dots, 20$ . It

holds that  $\sum_{k=1}^{20} m(I_k) = 1$ .





Shrinkage strain  $\epsilon_{sh}$ . Values *Bel* and *PI* for two sequences of probing intervals and the rich set of focal elements.

# Outline

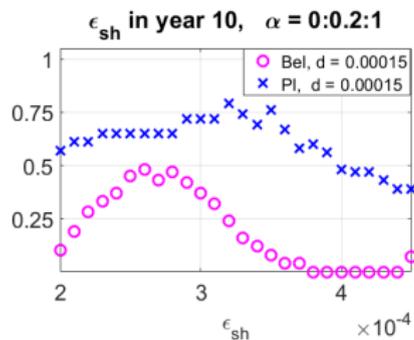
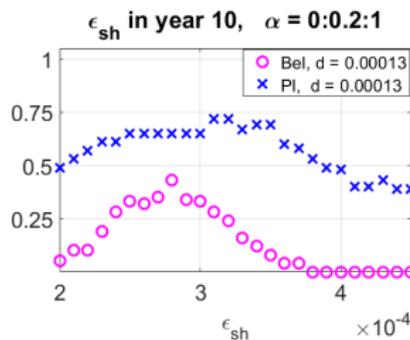
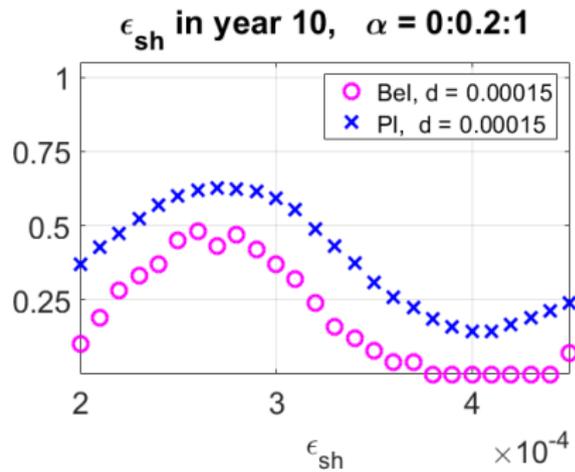
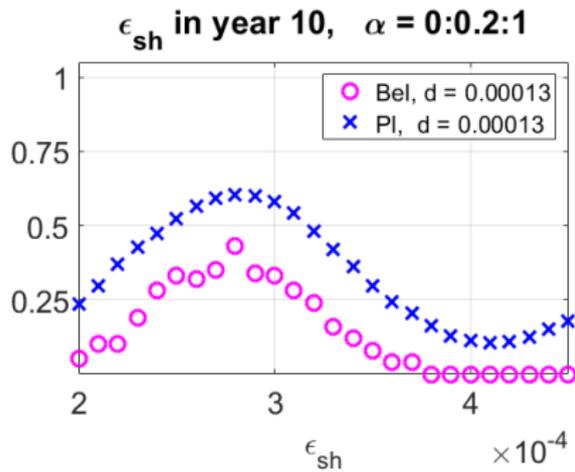
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# Coupling of fuzzy sets and the D-S theory IIB

Modification:

$$Pl_{\text{ratio}}(A) = \sum_{\text{all } A_i \cap A \neq \emptyset} m(A_i) \frac{\text{meas } A_i \cap A}{\text{meas } A_i}. \quad (5)$$

Shrinkage strain  $\epsilon_{sh}$ . Values  $Bel$  and  $Pl_{ratio}$ ; the rich set of focal elements.



# Conclusions

- Fuzzy scalars, vectors, functions, . . .
- Fuzzy set and D-S extension principles based on the worst/best scenarios.
- Optimization methods in searching for worst/best scenarios.
- Various couplings of fuzzy sets and the D-S theory offer a deeper insight into UQ results.

Thank you for your attention